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A SIXTH-ORDER DERIVATIVE-FREE ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract. *This article discusses a derivative-free iterative method to solve a nonlinear equation. The method derived by approximating derivatives on the method proposed by Rafiullah* [International Journal of Computer Mathematics, 4: 2459–2463, 2010] *using a central difference formula. We show analytically that the method has sixth-order of convergence. Numerical experiments show that the new method is comparable with other methods in terms of the speed in obtaining a root.*

1. INTRODUCTION

One of the well known topics discussed in the mathematical sciences is a technique to obtain the solution of nonlinear equation of the form

$$f(x) = 0. \tag{1}$$

The numerical method that can be used to solve (1) is based on iterative method. The famous iterative method appears in the literatures is Newton's method, given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \ldots. \tag{2}$$

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The method has quadratically convergence [4, p. 55]. Using Newton's method and Adomian's decomposition method [1, p. 10], Basto et al. [3] obtain a new iterative method of the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{f^2(x_n)f''(x_n)}{2f'(x_n)^3}\right), \tag{3}$$

which converges cubically [3].

Rafiullah [10] modifies (3) by estimating $f''(x_n)$ using a forward difference [6],

$$f''(x_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n}, \tag{4}$$

and combines the resulting equation with another Newton's method. He ends up with

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)(f'(x_n) - f'(y_n))}{2f'(x_n)^2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \tag{5}$$

Then, by estimating $f'(z_n)$ in (5) by a linear interpolation [8] based on two known points, namely $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$, and he obtains the following a sixth-order iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6}$$

$$z_n = y_n - \frac{f(x_n)(f'(x_n) - f'(y_n))}{2f'(x_n)^2}, \tag{7}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(x_n)}{4f'(x_n)f'(y_n) - f'(x_n)^2 - f'(y_n)^2}. \tag{8}$$

The rest of this paper is organized as follows. In section two, a new sixth order derivative-free iterative method is obtained by approximating the first derivative appear in (6)-(8) using central difference formulas [7, p. 313]. Then in section three the proposed method is tested on four test functions and compared with some known iterative methods.

2. A DERIVATIVE-FREE ITERATIVE METHOD

If we approximate $f'(x_n)$ and $f'(y_n)$ in (6)-(8) using a central difference, that is

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)} =: T_{1x}, \tag{9}$$

and

$$f'(y_n) \approx \frac{f(y_n + f(y_n)) - f(y_n - f(y_n))}{2f(y_n)} =: T_{1y}, \tag{10}$$

and substitute them into (6)-(8), we obtained the following new iterative method

$$y_n = x_n - \frac{f(x_n)}{T_{1x}}, \tag{11}$$

$$z_n = y_n - \frac{f(x_n)(T_{1x} - T_{1y})}{2(T_{1x})^2}, \tag{12}$$

$$x_{n+1} = z_n - \frac{2f(z_n)T_{1x}}{4T_{1x}T_{1y} - (T_{1x})^2 - (T_{1y})^2}. \tag{13}$$

In the following we show that the proposed method (11)–(13) is of order six.

Theorem 1.1 . *Let $f : D \subset \mathbb{R} \to \mathbb{R}$. Assume that $f \in C^6(D)$. If $f(x) = 0$ has a simple root at $\alpha \in D$ and x_0 is given and sufficiently close to α, then the new method defined by (11)–(13) is of order six and satisfies the following error equation*

$$e_{n+1} = -\frac{1}{4}\frac{c_2\left(c_1^6c_3^2 + 6c_1^4c_3^2 + 5c_3^2c_1^2 + 16c_2^2c_1c_3 - 16c_2^4\right)e_n^6}{c_1^5},$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!}$, $j = 1, 2, 3, \ldots$, and $e_n = x_n - \alpha$.

Proof. Substituting (9) into (11), we obtain

$$y_n = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}. \tag{14}$$

Let α be a simple root of $f(x) = 0$, then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Taylor's expansion of $f(x_n)$ about $x_n = \alpha$ is given by [2, p. 189]

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7), \tag{15}$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!}, j = 1, 2, 3, \ldots$ and $x_n - \alpha = e_n$. Using (15), we obtain respectively

$$\begin{aligned} 2f(x_n)^2 = 2c_1^2 e_n^2 + 4c_1c_2e_n^3 + \left(4c_1c_3 + 2c_2^2\right) e_n^4 + \left(4c_1c_4 + 4c_2c_3\right) e_n^5 \\ + (4c_1c_5 + 4c_2c_4 + 2c_3^2)e_n^6 + O(e_n^7), \end{aligned} \tag{16}$$

and

$$x_n + f(x_n) = e_n + \alpha + c_1e_n + c_2e_n^2 + c_3e_n + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7).$$

Taylor's expansion of $f(x_n+f(x_n))$ about $x_n+f(x_n) = \alpha$, and $f(x_n-f(x_n))$ about $x_n - f(x_n) = \alpha$ are given respectively by

$$\begin{aligned} f(x_n + f(x_n)) = {} & \left(c_1 + c_1^2\right) e_n + \left(3c_1c_2 + c_2c_1^2 + c_2\right) e_n^2 \\ & + \left(2c_2^2 + 3c_3c_1^2 + 4c_1c_3 + 2c_2^2c_1 + c_3 + c_3c_1^3\right) e_n^3 \\ & + \cdots + O(e_n^7), \end{aligned} \tag{17}$$

and

$$\begin{aligned} f(x_n - f(x_n)) = {} & \left(-c_1^2 + c_1\right) e_n + \left(-3c_1c_2 + c_2c_1^2 + c_2\right) e_n^2 \\ & + \left(-2c_2^2 + 3c_3c_1^2 - 4c_1c_3 + 2c_2^2c_1 + c_3 - c_3c_1^3 +\right) e_n^3 \\ & + \cdots + O(e_n^7). \end{aligned} \tag{18}$$

Subtracting (18) from (17) gives

$$\begin{aligned} f(x_n + f(x_n)) - f(x_n - f(x_n)) = {} & 2c_1^2e_n + 6c_1c_2e_n^2 + \left(2c_1^3c_3 + 8c_1c_3 + 4c_2^2\right) e_n^3 \\ & + \left(8c_1^3c_4 + 6c_1^2c_2c_3 + 10c_1c_4 + 10c_2c_3\right) e_n^4 \\ & + \cdots + O(e_n^7). \end{aligned} \tag{19}$$

Dividing (16) by (19) yields

$$\frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} = \frac{2c_1^2e_n^2 + 4c_1c_2e_n^3 + Ae_n^4 + Be_n^5 + Ce_n^6}{2c_1^2e_n + De_n^2 + Ee_n^3 + Fe_n^4 + Ge_n^5 + He_n^6}, \tag{20}$$

where

$$\begin{aligned} A &= 4c_1c_3 + 2c_2^2, \\ B &= 4c_1c_4 + 4c_2c_3, \\ C &= 4c_1c_5 + 4c_2c_4 + 2c_3^2, \\ D &= 6c_1c_2, \end{aligned}$$

$$
\begin{aligned}
E &= 2c_1^3c_3 + 8c_1c_3 + 4c_2^2,\\
F &= 8c_1^3c_4 + 6c_1^2c_2c_3 + 10c_1c_4 + 10c_2c_3,\\
G &= 2c_1^5c_5 + 20c_5c_1^3 + 24c_1^2c_2c_4 + 6c_3^2c_1^2 + 6c_1c_2^2c_3 + 12c_1c_5 + 12c_2c_4 + 6c_3^2,\\
H &= 12c_1^5c_6 + 10c_1^4c_2c_5 + 40c_6c_1^3 + 60c_5c_1^2c_2 + 30c_3c_1^2c_4 + 24c_1c_4c_2^2\\
&\quad + 12c_3^2c_1c_2 + 2c_2^3c_3 + 14c_1c_6 + 14c_2c_5 + 14c_3c_4.
\end{aligned}
$$

Simplifying (20) and applying a geometry series to the resulting equation, we end up with

$$\frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} = e_n + \left(-\frac{c_2}{c_1}\right) e_n^2 + \cdots + O(e_n^7). \tag{21}$$

Substituting (21) into (14), and noting $x_n = e_n + \alpha$, we obtain

$$y_n = \alpha + \left(-\frac{c_2}{c_1}\right) e_n^2 + \cdots + O(e_n^7). \tag{22}$$

Expanding $f(y_n)$ using Taylor's series about $y_n = \alpha$ and using (22) yield

$$f(y_n) = c_2e_n^2 + \left(\frac{-2c_2^2}{c_1} + c_3c_1^2 + 2c_3\right) e_n^3 + \cdots + O(e_n^7). \tag{23}$$

Computing (9) using (15) and (19), after simplifying, we have

$$T_{1x} = c_1 + 2c_2e_n + (c_1^2c_3 + 3c_3)e_n^2 + \cdots + O(e_n^7). \tag{24}$$

Similarly, we obtain

$$T_{1y} = c_1 + \left(\frac{2c_2^2}{c_1}\right) e_n^2 + \cdots + O(e_n^7). \tag{25}$$

Substituting (15), (23), (24), and (25) into (12), and after some algebra, we obtain

$$z_n = \alpha + \left(\frac{1}{2}\frac{c_3}{c_1} + \frac{2c_2^2}{c_1^2} + \frac{1}{2}c_1c_3\right) e_n^3 + \cdots + O(e_n^7). \tag{26}$$

By the same strategy to obtain $f(y_n)$, using (26) we obtain

$$f(z_n) = \left(\frac{2c_2^2}{c_1} + \frac{1}{2}c_3c_1^2 + \frac{1}{2}c_3\right) e_n^3 + \cdots + O(e_n^7). \tag{27}$$

Substituting (24), (25), (26), and (27) into (13), using geometric series and simplifying the resulting equations we ends up with

$$x_{n+1} = \alpha - \frac{1}{4}\frac{c_2\left(c_1^6c_3^2 + 6c_1^4c_3^2 + 5c_3^2c_1^2 + 16c_2^2c_1c_3 - 16c_2^4\right)e_n^6}{c_1^5}. \tag{28}$$

Since $e_{n+1} = x_{n+1} - \alpha$ then (28) becomes

$$e_{n+1} = -\frac{1}{4}\frac{c_2\left(c_1^6c_3^2 + 6c_1^4c_3^2 + 5c_3^2c_1^2 + 16c_2^2c_1c_3 - 16c_2^4\right)e_n^6}{c_1^5}.$$

From the definition of the order of convergence, we see that (11)–(13) is of order six $\square$.

3. NUMERICAL SIMULATION

In this section, we compare the number of iteration to obtain an approximated root for Rafiullah's Method (MR), equation (8), Second Derivative-Free Variant of Halley's Method (MH) [5], A Sixth-Order Iterative Method Free from Derivative (MP) [9] and the Derivative-Free Iterative Method (MT) given by (11)-(13) using four test functions. The computation was carried out using Maple 17. The criteria to stop the iteration are $|f(x_{n+1})| \leq Tol$, and $|x_k - x_{k-1}| \leq Tol$, where $Tol \leq 1.0\times10^{-50}$. The maximum iteration allowed is 100.

Table 1: Comparison the number of iterations of the discussed iterative methods

$f(x)$	x_0	The number of iterations				α
		MR	MH	MP	MT	
$f_1 = e^{-x} + \cos(x)$	1.2	3	3	3	3	1.7461395304080124
	1.5	3	3	3	3	
	1.8	2	2	2	2	
	2.0	3	3	3	3	
	2.3	3	3	3	3	
$f_2 = x - 2 - e^{-x}$	0.5	3	2	3	3	2.120028238987641
	1.1	3	3	3	3	
	1.5	3	3	3	3	
	3.0	3	3	3	3	
	3.2	3	3	3	3	
$f_3 = \sqrt{x} - x$	0.5	3	3	3	3	1.000000000000000
	0.7	3	3	3	3	
	1.2	3	3	3	3	
	1.9	3	3	3	3	
	2.2	3	3	3	3	
$f_4 = xe^{-x} - 0.1$	-1.2	7	4	*	4	0.111832559158963
	-0.6	4	3	4	4	
	-0.1	3	3	3	4	
	0.0	3	3	3	3	
	0.2	3	3	3	3	

Table 1 shows the number of iterations needed to obtain the approximated root for several mention methods by varying an initial guess x_0. The star mark (*) indicates that the method converges to a different root. However, there is no significant difference among mention methods in terms of the number of iteration needed to obtain an approximate root.

The simulation shows that MH is sightly better than other methods, however this method is not derivative free. In general MT is comparable to other methods, therefore this method can be used as an alternative method for solving nonlinear equation.

REFERENCES

1. G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, New York, 1986.
2. R. G. Bartle dan D. R. Shebert. *Introduction to Real Analysis, 4^{st} Ed*, John Wiley & Sons, Inc., New York, 1999.

3. M. Basto, V. Semiao dan F. L. Calheiros, *A new iterative method to compute nonlinear equation*, Applied Mathematics and Computation, 173 (2006), 468–483.
4. J. D. Faires and R. L. Burden, *Numerical Methods, 3^{rd} Ed*, Brooks Cole, Belmont, 2002.
5. J. Han, H. He, A. XU and Z. Cen, *A second derivative free variant of Halley's method with sixth orde convergence*, Journal of Pure and Applied Mathematics: Advances and Applications, 3 (2011), 195–203.
6. J. Kou, Y. Li, and X. Wang, *Modied Halley's method free from second derivative*, Applied Mathematics and Computation, 183 (2006), 704–708.
7. J. H. Mathews and K. D. Fink, *Numerical Methods Using MATLAB, 3^{rd} Ed.* Prentice Hall, New Jersey, 1999.
8. S. K. Parhi and D. K. Gupta, *A six order method for nonlinear equation*, Applied Mathematics and Computation, 203 (2008), 50–55.
9. R. Qudsi, M. Imran and Syamsudhuha, *A sixth-order iterative method free from derivative for solving multiple roots of a nonlinear equation*, Applied Mathematical Sciences, 8 (2014), 5721–5730.
10. M. Rafiullah, *Three-step iterative method with sixth order convergence for solving nonlinear equations*, International Journal of Computer Mathematics, 4 (2010), 2459–2463.

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